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# The nature of the $S$ dependence of spin traces

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**Abstract.** Using the quasiclassical formalism we present a simple proof that traces of products of $N$ spin operators are polynomials in $S(S+1)$ of degree $[N/2]$.

In mathematical and theoretical physics, traces of functions of spin operators often occur. A prime example is the statistical mechanics of (quantum) Heisenberg systems (Heisenberg 1928) where the free energy and the correlation functions are related directly to such traces.

Given the spin, $S$, the spin operators $\boldsymbol{S}_i$ ($i=1, 2, 3$) are the three generators of O(3) in the $(2S+1)$-dimensional representation. Then traces of a product of $N$ such operators

$$F_{i_1 \dots i_N} \equiv \mathrm{Tr}[\boldsymbol{S}_{i_1} \dots \boldsymbol{S}_{i_N}]/(2S+1) \tag{1}$$

are invariant tensors of O(3). Since there are only two independent invariants, $\delta_{ij}$ and $\varepsilon_{ijk}$, the *tensorial* nature of $F$ can be nothing but products of these two objects. However, the dependence of $F$ on $S$ is not completely transparent. Experience and intuition both point to the fact that $F$ depends on $S$ only through the parameter $S(S+1)$. Because this is the 'value' of the Casimir $\boldsymbol{S}\,.\,\boldsymbol{S}$ and $\delta_{ij}$ has intimate links with it, one could reasonably expect this parameter to emerge. In this brief paper, we employ the quasiclassical formalism (Kaplan 1971, Mirkovitch and Summerfield 1973 and earlier references therein), which lends itself to this question naturally, to obtain a simple proof of the nature of this dependence.

*Preliminaries.* We will need a trivial lemma

$$F_{i_1 \dots i_N} = F^*{}_{i_N \dots i_1} \tag{2}$$

which follows from ($a$) $\mathrm{Tr}M = \mathrm{Tr}M^T$ and ($b$) the hermiticity of $\boldsymbol{S}_i$. Next, let us review briefly the relevant parts of the quasiclassical approach. To each spin operator we associate its Wigner equivalent:

$$\boldsymbol{S}_i \rightarrow (\boldsymbol{S}_i)_{\mathrm{W}} = S\Omega_i$$

where $\Omega_i$ are the components of a unit vector in ordinary Euclidean three-space. (The unit matrix is identified with the function 1.) The algebra satisfied by these spins is then 'taken care of' by an operator (Groenewold 1946, Kaplan 1971) in the function space of $\Omega$. In particular, products of spins are represented by

$$\boldsymbol{S}_{i_1}\boldsymbol{S}_{i_2} \dots \boldsymbol{S}_{i_N} \rightarrow (S\Omega_{i_1})G[(S\Omega_{i_2})G[\dots[(S\Omega_{i_N})G(1)]\dots] \tag{3}$$

with

$$G = 1 + \frac{1}{2S}\overleftarrow{\partial}_a[\delta_{ab} - \Omega_a\Omega_b + \mathrm{i}\varepsilon_{abc}\Omega_c]\overrightarrow{\partial}_b + \dots. \tag{4}$$

Refering the reader to Kaplan (1971) for further details, let us just comment that

(i) the differential operators satisfy $\partial_a\Omega_b = \delta_{ab}$;

(ii) the arrows on them indicate the direction they operate;

(iii) the brackets indicate that $\overrightarrow{\partial}$ operates on *everything* to its right while $\overleftarrow{\partial}$ operates only on the $\Omega$ to its left;

(iv) the ... represent higher derivatives which do not enter in (3) since $\Omega\overleftarrow{\partial}^p \equiv 0$ for $p > 1$. It is convenient to multiply (3) by $2^N$ and express the RHS of (3) as

$$\vec{H}_{i_1}\vec{H}_{i_2}\dots\vec{H}_{i_N}(1) \tag{5}$$

where

$$\vec{H}_i \equiv 2S\Omega_i G = 2J\Omega_i + \vec{O}_i. \tag{6}$$

For later convenience, we have defined

$$J \equiv S + \tfrac{1}{2} \tag{7}$$

and

$$\vec{O}_i \equiv -\Omega_i + [\delta_{ib} - \Omega_i\Omega_b + \mathrm{i}\epsilon_{ibc}\Omega_c]\overrightarrow{\partial}_b. \tag{8}$$

In this formalism, the trace operation is just $(2S+1)\int \mathrm{d}\Omega/(4\pi)$, the integration over the 'surface' of the sphere. Thus

$$F \equiv \mathrm{Tr}[\boldsymbol{S}_{i_1}\dots\boldsymbol{S}_{i_N}]/(2S+1) = 2^{-N}\int (\mathrm{d}\Omega/4\pi)\vec{H}_{i_1}\dots\vec{H}_{i_N}(1). \tag{9}$$

Finally, the last 'preliminary' concerns the possibility of integration by parts. In (9), there are lots of derivatives acting to the right. We can change them to actions on the left, though this procedure is *not trivial.* In the appendix, we show that, under the integral, $\vec{H}_i$ is the same as $\overleftarrow{H}_i$ with

$$\overleftarrow{H}_i \equiv 2J\Omega_i - \overleftarrow{O}_i \tag{10}$$

where

$$\overleftarrow{O}_i \equiv \overleftarrow{\partial}_b[\delta_{ib} - \Omega_i\Omega_b + \mathrm{i}\epsilon_{ibc}\Omega_c] - \Omega_i. \tag{11}$$

Now we are ready to prove the following theorem.

*Theorem.* $F$ is a polynomial in $S(S+1)$ of degree $[N/2]$, the largest integer in $N/2$.

*Proof.* In terms of $J$, this theorem may be restated as: $F$ is a polynomial in $J^2$ with $N$ as the highest possible power in $J$. Since $\vec{H}$ is linear in $J$, equation (9) immediately tells us that $F$ is a polynomial of degree $N$ in $J$. We only need to prove that $F$ is even in $J$, thus eliminating all the odd powers.

Consider

$$(4\pi)2^N F_{i_1\dots i_N}(J) = \int \mathrm{d}\Omega(2J\Omega + \vec{O})_{i_1}\dots(2J\Omega + \vec{O})_{i_N} \tag{12a}$$

$$= \int \mathrm{d}\Omega(2J\Omega - \overleftarrow{O})_{i_1}\dots(2J\Omega - \overleftarrow{O})_{i_N} \tag{12b}$$

$$= \int \mathrm{d}\Omega(-2J\Omega + \overleftarrow{O}^*)_{i_1}\dots(-2J\Omega + \overleftarrow{O}^*)i_N. \tag{12c}$$

Integration by parts led to (12*b*). To obtain (12*c*), we let all $\Omega \to -\Omega$. All parts of $\tilde{O}$ are odd except the $\tilde{\partial} i \epsilon \Omega$ term; this is the reason for the complex conjugation. But (12*c*) is nothing but $2^N F^*_{i_N \ldots i_1}(-J)$. Using (2), we arrive at the evenness of *F*, QED.

We close with several remarks.

(1) If a function of spin operators can be expressed in a (convergent) expansion in powers of **S**, then the trace of that function is also even in *J*. An example is $\exp(\boldsymbol{s} \cdot \boldsymbol{t})$.

(2) If we were interested only in the *F*'s which are totally symmetric in the indices, then we could employ the formula

$$\exp(B_i S_i) \to \exp[B_i(S\Omega_i G)](1) = (\cosh B + \Omega_i \sinh B_i)^{2S}. \tag{13}$$

The integral over $\Omega$ is easily performed to give

$$\left(1 + \frac{1}{3!}(2S+1)^2 B^2 + \ldots\right)\left(1 + \frac{1}{3!}B^2 + \frac{1}{5!}B^4 + \ldots\right)^{-1}. \tag{14}$$

Since $(2S+1) = 2J$, the evenness of *J* is explicit. Expansion of (14) in powers of $B_i$ will produce the totally symmetric part of *F*. Clearly, only even *N* contribute to these tensors—any odd *N* must involve at least one $\varepsilon_{ijk}$. The application of this result is limited, however, since the *B*'s of the most interesting problem—the Heisenberg system—do not commute.

(3) One immediate application of this theorem is the following. In a $1/S$ expansion of the Heisenberg system (Harrigan and Jones 1973) about the classical limit ($S = \infty$), every other power of the expansion requires no more calculation than substituting $[S(S+1)]^{1/2}$ into *S*. This confusing statement is much more transparent in the language of *J*. Consider a function of *J* obtained by traces, $Z(J)$. According to this theorem, *Z* is a function of $J^2$, so that

$$Z = Z_0 + J^{-2} Z_1 + J^{-4} Z_2 + \ldots.$$

In terms of *S*, this is just

$$Z_0 + S^{-2} Z_1 + S^{-3}(-Z_1) + S^{-4}(Z_2 + \tfrac{3}{4} Z_1) + S^{-5}(-2Z_2 - \tfrac{1}{2} Z_1) + \ldots,$$

from which we see that only every other order in *S* produces new corrections.

(4) Work is in progress to generalise the quasiclassical formalism to other Lie groups. Typically, more than one independent Casimir exists and several parameters are required to label a representation. It would be interesting to see how this theorem manifests itself in those cases.

## Appendix

We wish to show that

$$\int \mathrm{d}\Omega\, f\vec{H} g = \int \mathrm{d}\Omega\, f\tilde{\vec{H}} g \tag{A.1}$$

where *f* and *g* are arbitrary functions of $\Omega$.

To facilitate our work, we use the equivalent representation for $\vec{H}$ (Chang *et al* 1971)

$$\vec{H}_a \equiv 2S\Omega_a \vec{G} = 2S\Omega_a - \Omega_a \vec{L}_i \vec{L}_i + \mathrm{i}\Omega_a \epsilon_{ijk} \vec{L}_i \Omega_j \vec{L}_k \tag{A.2}$$

where

$$L_a = -\mathrm{i}\epsilon_{abc}X_b\nabla_c \qquad \Omega_a = X_a/r. \tag{A.3}$$

It is clear that

$$L_a\Omega_b = \mathrm{i}\epsilon_{abc}\Omega_c \tag{A.4}$$

and

$$(L_aL_a)\Omega_b = 2\Omega_b. \tag{A.5}$$

To connect $L_a$ with $\partial_b$, we note

$$L_a = \mathrm{i}\epsilon_{abc}\Omega_c\partial_b. \tag{A.6}$$

Integration by parts with $\vec{L}$ is just

$$\int \mathrm{d}\Omega f\vec{L}g = -\int \mathrm{d}\Omega f\vec{L}g. \tag{A.7}$$

Using the relation on the $\vec{L}$ of the $\vec{H}$, we see that

$$\int f\vec{H}_a g = \int f\{2S\Omega_a + (\Omega_a\vec{L}_i\vec{L}_i) - \mathrm{i}\epsilon[(\Omega\vec{L}\Omega)\vec{L}]\}g - \int f\{\vec{L}_i\Omega_a\vec{L}_i) - \mathrm{i}\epsilon\vec{L}(\Omega\vec{L}\Omega)\}g.$$

But

$$-\mathrm{i}\epsilon_{ijk}(\Omega_a\vec{L}_i\Omega_j)\vec{L}_k = (\epsilon_{ijk}\epsilon_{iab}\Omega_b\Omega_j)\vec{L}_k \equiv 0.$$

So, using (A.5), we have

$$\begin{aligned}\int f\vec{H}_a g &= \int f\{(2S+1)\Omega_a + [\Omega_a - \vec{L}_i(\vec{L}_i\Omega_a) + \mathrm{i}\epsilon_{ijk}\vec{L}_k(\Omega_j\vec{L}_i\Omega_a)]\}g \\ &= \int f\{2J\Omega_a - \vec{O}_a\}g\end{aligned}$$

The operator in the curly brackets is *defined* to be $\vec{H}$.

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